

# Systematic unification of the partition functions of various field theories in a two-dimensional classical $\phi^4$ model

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## Abstract

We show that the partition functions of several field theories with different symmetries and in various dimensions are equal to the partition function of a two-dimensional (2D) classical  $\phi^4$  field theory model. Although the completeness of 2D  $\phi^4$  model has been already proved by introducing a quantum formalism for the partition functions, in this paper we study it in a fundamental systematic approach with purely mathematical basis to re-derive such completeness in a general manner. Due to generality, such re-derivation will lead to more clarification of the result for other communities of physics as well as quantum information theorists. Furthermore, our proof of the completeness has an important advantage that is a step-by-step proof which is supported by some graphical transformations. Specifically, we consider discrete version of field theories on lattices with arbitrary dimensions where they are defined as arbitrary polynomial functions of fields with different symmetries. Then we give a set of simple graphical transformations on such field theories to convert them to a classical  $\phi^4$  field theory on a 2D square lattice with the same partition function.

## 1 Introduction

One of the greatest of ambitions in physics is to find a single theory where one can unify all the existing theories [1]. This problem especially is more challenging for particle physicists. They made many attempts to unify all fundamental interactions in a unique theory leading to the standard model of elementary particles[2]. A similar idea also has been considered in other branches of theoretical sciences, such as computer science. There are some difficult problems which are called NP-complete where if one solves one of the NP-complete problems [3], one will be able to solve all other problems in computer science. For example, each problem in computer science can be converted into a version of the SAT problem [4].

Moreover the above fundamental sciences, it has also been considered a new kind of the unification in statistical mechanics which has been called completeness. Since all thermodynamics properties of a model can be derived from the partition function, the main idea is related to unification of the partition functions of various statistical mechanical models in the partition function of a complete model. It means that there is a specific model Hamiltonian in the sense that the partition function of such a

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model in different regimes of coupling constants can generate the partition functions of other statistical mechanical models.

The first complete model was introduced in an interesting paper [5] where the authors showed that the partition function of the 2D Ising model with complex interactions and magnetic field is equal to the partition function of any statistical mechanical models. It means that one can imagine the phase diagram of a 2D Ising model with the magnetic field where it covers all universality classes.

Although the completeness of 2D Ising model is related to statistical physics, quantum information theory plays a key role for the derivation of the result. It is shown that the partition function of a spin model can be written as the inner product of two quantum states [6, 7] in the following form:

$$\mathbf{Z} = \langle \alpha | G \rangle \quad (1)$$

where  $|\alpha\rangle$  is a product state that encodes the couplings of the spin models and  $|G\rangle$  is an entangled state that encodes the interaction pattern of the spin model. By quantum formalism, the completeness of 2D Ising model is related to the universality of cluster states for measurement-based quantum computation [8, 9, 10]. Furthermore, a similar approach leads to other complete models like 4D  $Z_2$  lattice gauge theory [11, 12]. An extension of above results to statistical models with continuous degrees of freedom has been lead to completeness of 4D  $U(1)$  lattice gauge theory [13] and 2D classical  $\phi^4$  field theory [14].

In addition to completeness, mapping between quantum information theory and statistical mechanics has also opened other interesting fields of research [15, 16, 17, 18, 19, 20, 21]. However, quantum information theory usually plays the role of a middle approach for derivation of results and it is expected to find an alternative approach to can be accessible to a wider range of physicists. Furthermore, such new approaches will lead to clarification of the results for quantum information theorists. For this purpose, in [22], authors gave an algorithmic proof for the completeness of 2D Ising model without quantum formalism. Furthermore in a recent paper [23], the new concept of the universal models has been introduced where, by a bridging to computer science, it has been shown that 2D Ising model in presence of magnetic field with real coupling constants is a universal model. It means that the physics of all classical models with discrete or continuous degrees of freedom are reproduced in the low-energy sector of a 2D Ising model. By such a definition, a complete model is also a specific kind of a universal model.

In spite of the above improvements on completeness and universality of spin models, there are few results for statistical models with continuous variables. Completeness of 2D classical  $\phi^4$  field theory and 4D  $U(1)$  lattice gauge theory has been proved by quantum formalism and alternative approaches will lead to better understanding the results for other physical communities.

Following such improvements, we systematically consider completeness of 2D  $\phi^4$  model by a purely mathematical method. In addition to theoretical importance, we believe that our work has two other more advantages, simplicity and generality where it will clarify the previous result that had been derived by quantum formalism for other communities of physics as well as quantum information theorists. Furthermore, our approach is algorithmic in nature where we consider a general field theory with arbitrary symmetries and arbitrary dimensions and we give a set of specific transformations on such models to convert them to a 2D  $\phi^4$  field theory with the same partition function. We also display all transformations by some simple graphical notations so that one can graphically perform all steps of conversion to 2D  $\phi^4$  model. In this way, we provide a useful method that can be used to explicitly find 2D  $\phi^4$  model corresponding to the other field theories.

The structure of this paper is as follows: In section (2) we briefly review the definitions of the general classical field theories and also  $U(1)$  lattice gauge theories. In section (3) we give a simple transformation of the partition function of the above models to convert them to a unique form with a simple kinetic term in the Hamiltonian. In section (4), we consider interacting terms of fields which can be

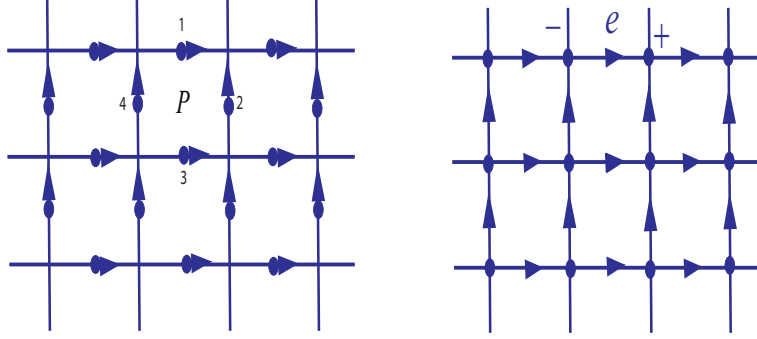


Figure 1: (Color online) Left: An oriented lattice for  $U(1)$  lattice gauge theory where field variables live in edges of the lattice. A plaquette of the lattice include four field variables which are denoted by numbers 1 to 4. right: An oriented lattice for classical field theory where field variables live in vertices of the lattice. Two variables in two endpoints of each edge are denoted by  $+$  and  $-$ .

arbitrary polynomial functions and give a transformation to reduce each polynomial function to a  $\phi^4$  term. In section (5), we consider how to decrease dimensions of initial models to two and then in section (6), we give the last transformations to match completely the model to a 2D square lattice. Finally, in section (7) we show the completeness of 2D  $\phi^4$  field theory and discuss on the efficiency of our transformations. We emphasize that the partition function of initial models stay invariant under all transformations that we apply.

## 2 A brief review of classical field theories and $U(1)$ lattice gauge theories

In this section, we briefly review the definitions of two categories of field theories, classical field theories and  $U(1)$  lattice gauge theories. These models belong to two important symmetrical categories containing global and local symmetries.

Classical field theories have important theoretical applications in different fields of physics. On the one hand, they are important models for studying different ideas in statistical mechanics[24, 25, 26, 27] and on the other hands they are useful for studying high-temperature behavior of quantum field theories [32, 33] and for studying non-perturbative dynamics of low-momentum models in nonequilibrium quantum field theory [28, 29, 30, 31]. In this paper, we will deal with a general form of classical field theories and will study some mathematical properties of their partition functions.

A classical field theory is defined by a Hamiltonian which can be a polynomial function of fields and derivations of fields in the following general form:

$$H = \int D\phi h(\partial\phi, \partial^2\phi, \dots, (\partial\phi)^2, \dots, \phi, \phi^2, \dots, \phi\partial\phi, \dots). \quad (2)$$

Here we consider only a specific form of such models which are more well-known in the following form:

$$H = - \int D\phi \{V(\partial\phi) + W(\phi)\}, \quad (3)$$

where  $V(\partial\phi)$  ( $W(\phi)$ ) can be a polynomial function of  $\partial\phi$  ( $\phi$ ) like  $V(\partial\phi) = (\partial\phi)^2$ . Although such a model is defined on the continuous spacetime, there is also a discrete version of that on the discrete

spacetime. To this end, the continuous spacetime is approximated by a lattice where continuous variables  $\phi_i$  live in the vertices of the lattice, see Figure (1, right) as a sample lattice. After discretization, the Hamiltonian is converted to a new one on an oriented lattice in the following form:

$$H = - \sum_e V_e(\phi_+ - \phi_-) - \sum_i W_i(\phi_i), \quad (4)$$

where  $e$  refers to the edges of the lattice and  $\phi_+$  and  $\phi_-$  refer to two variables which live in two end-points of the edge  $e$ , see Figure(1).  $V_e$  ( $W_i$ ) refers to an arbitrary polynomial function, and it can be in different forms for various edges (vertices) of the lattice. We call  $V_e$  and  $W_i$  edge term and vertex term of the Hamiltonian respectively.

Another important field theories with local symmetries are  $U(1)$  lattice gauge theories which are known as a discrete version of electrodynamics [34] and are also generalizations of Wegner's Ising gauge theories [35, 36, 37, 38]. While there are two different kinds of such models which are called compact and non-compact lattice gauge theories [39, 40, 41], in this paper, we will deal with the compact one where the degrees of freedom are directly elements of  $U(1)$  group.

A  $U(1)$  lattice gauge theory is defined on a rectangular lattice which is considered as the discretized spacetime in the arbitrary dimension where continuous variables live in edges of the lattice, see Figure (1, left) as a sample lattice. For arbitrary directions on all edges of the lattice like Figure(1, left), the Hamiltonian of this model is defined in the following form:

$$H = - \sum_p J_p \cos(\phi_1 - \phi_2 - \phi_3 + \phi_4), \quad (5)$$

where  $J_p$  refers to coupling constants corresponding to each plaquette of the lattice which is denoted by "p" and sign of variables  $\phi_i$  is chosen according to directions of corresponding edges when we traverse a plaquette in the clockwise direction.

### 3 Transformation to a simple kinetic term

In this section, we consider the partition function of discrete version of the classical field theories and  $U(1)$  lattice gauge theories which is expressed as an integral on field variables. We apply a changing variable on the integral relation of the partition function. We show that such changing variables lead to a new model with the same partition function. In this way, we show that a large set of field theories and also  $U(1)$  lattice gauge theories can be unified in a new model with a simple kinetic term.

At the first consider the discrete version of a classical field model on an arbitrary lattice with  $N$  vertices where continuous variables  $\phi_i$  live in the vertices of the lattice in the form of (4). The partition function of such a model is written in the following form:

$$\mathbb{Z} = \int D\phi e^{\sum_e V_e(\phi_+ - \phi_-) + \sum_i W_i(\phi_i)}, \quad (6)$$

where  $D\phi$  refers to product of  $d\phi_1 d\phi_2 \dots d\phi_N$  and we absorb coefficient  $\beta = \frac{1}{k_B T}$  in the Hamiltonian for all relation in this paper.

In the first step, consider an edge of the lattice which is denoted by  $e$ . We define new continuous variables on any edges of the lattice as  $\psi_e = \phi_+ - \phi_-$  and replace them in the partition function in relation (6). Due to this change of variable, it is necessary to add many delta functions in the partition function. In fact, new variables  $\psi_e$  are not independent variables, and there are many constraints on them. Since  $\psi_e = \phi_+ - \phi_-$  we conclude that  $\phi_+ - \psi_e - \phi_- = 0$ . We can apply such a constraint in

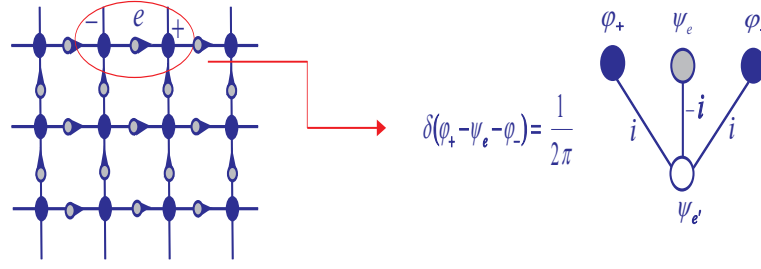


Figure 2: (Color online) Left: New field variables are inserted in any edges of the lattice and are denoted by gray circles. Right: A constraint between the edge variable  $\psi_e$  and vertex variables  $\psi_+$  and  $\psi_-$  is applied by a delta function which is equal to the partition function of a model corresponding to right-hand graph .

the partition function relation by an integral of a delta function in the form of  $\int d\psi_e \delta(\phi_+ - \psi_e - \phi_-)$ . It is clear that many constraints are corresponding to all edges of the graph where after the change of variables, many delta functions would add to satisfy the constraints. Therefore the partition function in relation (6) is converted to a new one in the following form:

$$\mathbb{Z} = \int D\psi D\phi e^{\sum_e V_e(\psi_e) + \sum_i W_i(\phi_i)} \prod_e \delta(\phi_+ - \psi_e - \phi_-), \quad (7)$$

where  $D\psi$  refers to product of  $\prod_e d\psi_e$ .

In the next step we use a simple relation for the definition of a delta function in the following form:

$$\delta(\phi_+ - \psi_e - \phi_-) = \frac{1}{2\pi} \int e^{i\psi_{e'}(\phi_+ - \psi_e - \phi_-)} d\psi_{e'}, \quad (8)$$

where we define new continuous variables  $\psi_{e'}$  corresponding to all edges of the lattice. This relation is graphically shown in Figure (2). If we use the above relation instead of delta functions in the partition function, we will have:

$$\mathbb{Z} = \int D\psi D\phi e^{\sum_e V_e(\psi_e) + \sum_i W_i(\phi_i)} \prod_e \int d\psi_{e'} e^{i\psi_{e'}(\phi_+ - \psi_e - \phi_-)}. \quad (9)$$

And after re-writing the partition function, it will be in the following form:

$$\mathbb{Z} = \frac{1}{(2\pi)^E} \int \prod d\psi_e \prod d\phi_i \prod d\psi_{e'} e^{i \sum_e \psi_{e'}(\phi_+ - \psi_e - \phi_-) + \sum_e V_e(\psi_e) + \sum_i W_i(\phi_i)}, \quad (10)$$

where  $E$  is the number of edges of the lattice. The above relation shows that the partition function of the initial model on the initial lattice is converted to the partition function of a new model on a new lattice with more variables where we should add new variables  $\psi_e$  and  $\psi_{e'}$  corresponding to all edges of the lattice, see Figure (2). If we apply such transformations corresponding to all edges of the initial lattice, we will have a new lattice where the partition function of new model corresponds to a Hamiltonian in the following general form:

$$H = \sum_{e,i} \pm i\psi_{e'}\phi_i + i\psi_{e'}\psi_e - \sum_i W_i(\phi_i) - \sum_e V_e(\psi_e), \quad (11)$$

where  $+i(-i)$  corresponds to vertex variable which was already denoted by  $\phi_+(\phi_-)$ . If we use a simple identity that  $ixy = -\frac{i}{2}[(x-y)^2 - x^2 - y^2]$ , we can conclude that the new Hamiltonian is a new

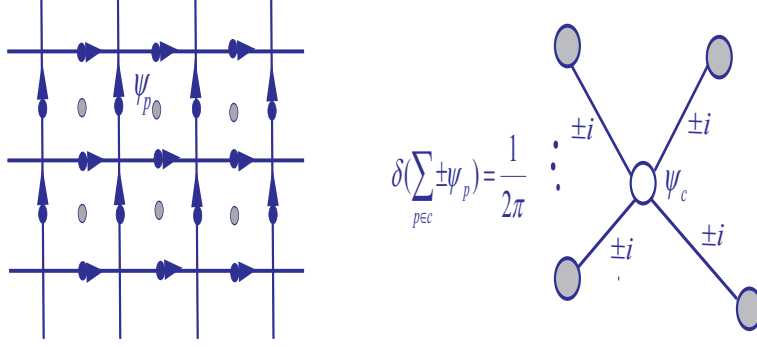


Figure 3: (Color online) Left: New field variables are inserted in any plaquettes of the lattice and are denoted by gray circles. Right: Related to structure of the lattice there are constraints between new plaquette variables  $\psi_p$  which are applied by delta functions. Each delta function as  $\delta(\sum \pm \psi_p)$  is equal to the partition function of a model corresponding to right-hand graph.

field theory which has a simple kinetic term as  $\frac{i}{2}(\phi_i - \phi_j)^2$  instead of the edge terms  $V_e(\phi_+ - \phi_-)$  of the initial model. Moreover, the edge term of the initial Hamiltonian  $V_e(\phi_+ - \phi_-)$  are added to vertex terms of the new model.

We can also repeat the above process for a lattice gauge theory. To this end, consider an arbitrary oriented lattice where variables  $\phi_i$  live on the edges of that and the Hamiltonian is in the form of (5). The partition function of such a model is written in the following form:

$$\mathbb{Z} = \int D\phi e^{\sum_p J_p \cos(\phi_1 - \phi_2 - \phi_3 + \phi_4)}. \quad (12)$$

We define plaquette variables  $\psi_p = \phi_1 - \phi_2 - \phi_3 + \phi_4$  corresponding to all plaquettes of the lattice, see Figure (3, left). It is clear that the new variables  $\psi_p$  are not independent and we should consider many constraints which generally are written as  $\sum_{p \in c} \pm \psi_p = 0$  where sign of field variables in this relation and form of the constraint depends on structure of the lattice. Such constraints can be applied to the partition function by many delta functions and we will have:

$$\mathbb{Z} = \int D\psi e^{\sum_p J_p \cos(\psi_p)} \prod_c \delta(\sum_{p \in c} \pm \psi_p). \quad (13)$$

As an example in Figure (4-a), we show a cubic of a three-dimensional lattice corresponding to a 3D  $U(1)$  lattice gauge theory where field variables  $\phi_1, \dots, \phi_{12}$  live in the edges of that cubic. As it is shown in Figure (4-b), we define six plaquette variables  $\psi_1, \dots, \psi_6$  corresponding to six plaquettes of the cubic. By considering the definition of each plaquette variable  $\psi_p$  in term of variables  $\phi_i$ , it is simple to check that a relation as  $\psi_1 - \psi_2 - \psi_3 + \psi_4 - \psi_5 + \psi_6 = 0$  holds. Finally, we should apply this constraint to the partition function by a delta function as  $\delta(\psi_1 - \psi_2 - \psi_3 + \psi_4 - \psi_5 + \psi_6)$  and then we convert it to an integral form by adding a new variable  $\psi_c$ , see Figure (4-c) for a graphical notation.

Generally for a lattice gauge theory with an arbitrary dimension, we can apply the above process and we have the following relation for the partition function by replacing all delta functions as exponential forms, see Figure (3, right) for a graphical notation of the delta functions:

$$\mathbb{Z} = \int D\psi e^{\sum_p J_p \cos(\psi_p)} \prod_c \frac{1}{2\pi} \int d\psi_c e^{i\psi_c (\sum_{p \in c} \pm \psi_p)}, \quad (14)$$

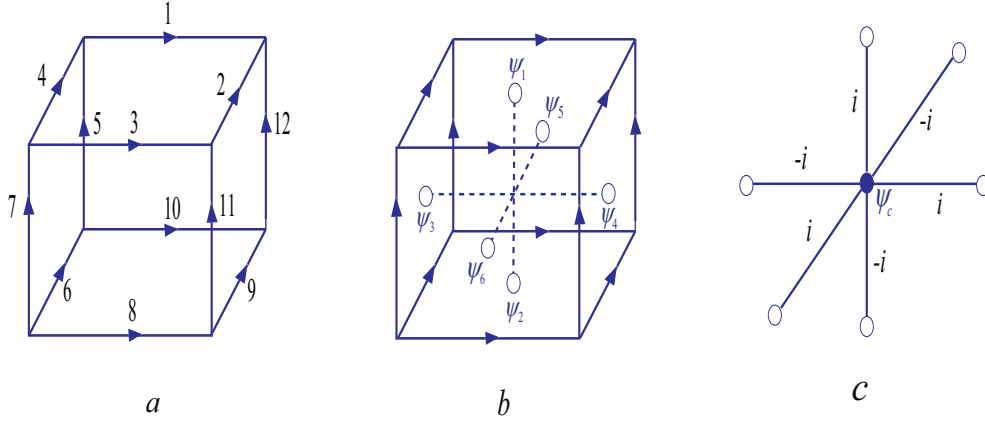


Figure 4: (Color online) a) a qubic of a three-dimensional lattice which is related to a 3D  $U(1)$  lattice gauge theory. The field variables live in the edges of the qubic. b) The plaquette variables are defined on any plaquettes of the qubic which are denoted by circles, as an example  $\psi_2 = \phi_6 - \phi_8 - \phi_9 + \phi_{10}$ . c) The constraint between plaquette variables as  $\psi_1 - \psi_2 - \psi_3 + \psi_4 - \psi_5 + \psi_6 = 0$  is applied to the partition function by adding a new variable  $\psi_c$  which is denoted by blue circle.

where we define new variables  $\psi_c$  corresponding to each constraint which is denoted by "c". If we apply transformations corresponding to all constraints in the partition function, it will be converted to a new model on a new lattice and the Hamiltonian will be in the following form:

$$H = \sum_c \sum_{p \in c} \pm i \psi_c \psi_p - \sum_p J_p \text{Cos}(\psi_p). \quad (15)$$

Therefore similar to the classical field model, a  $U(1)$  lattice gauge theory is also transformed to a classical field model with a kinetic term as  $\frac{i}{2}(\psi_c - \psi_p)^2$  and the plaquette terms  $\text{Cos}(\phi_1 - \phi_2 - \phi_3 + \phi_4)$  of the initial model are converted to vertex terms in the new model.

We can summarize the result of this section as the following message: The partition function of general classical field theories and  $U(1)$  lattice gauge theories on arbitrary graphs are equal to the partition function of a classical field theory with a simple kinetic term as  $\frac{i}{2}(\phi_i - \phi_j)^2$  on a different graph. This result is the beginning of transformations that we will follow in next sections. Specially we emphasize that the Hamiltonian of the new model is still very general so that it has been defined on a complex graph and vertex terms of the new Hamiltonian are polynomial functions as  $V(\phi)$ ,  $W(\phi)$  and  $\text{Cos}(\phi)$ . In the next section we go toward more specification of the model by reduction of polynomial functions to a  $\phi^4$  term.

## 4 Reduction to $\phi^4$ term

In the previous section, we showed that the partition function of a classical field theory with a simple kinetic term in the following form is equal to the partition function of a broad set of other classical field theories and  $U(1)$  lattice gauge theories:

$$H = \sum_{\langle i,j \rangle} \pm i \phi_i \phi_j - \sum_i P_i(\phi_i) \quad (16)$$

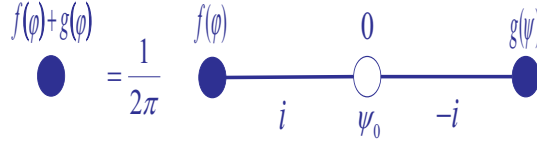


Figure 5: (Color online) A field variable can be splitted to two new variables where corresponding polynomial function  $P$  is also splitted to functions  $f$  and  $g$ . This rule changes the partition function by a factor  $\frac{1}{2\pi}$ .

where this model is defined on a complex graph and  $\langle i, j \rangle$  refers to neighboring vertices of that graph. For brevity, we denoted all variables by initial form  $\phi_i$  and  $P_i$  refers to a polynomial function which can be in different forms for various vertices.

Due to polynomial functions  $P_i$ , the above Hamiltonian is still very general and in this section we define an important transformation in the relation of the partition function to reduce the polynomial functions to a  $\phi^4$  term. To this end, we give a simple lemma in the following form:

**Lemma:** If the polynomial function  $P$  is a summation of two functions as  $P(\phi) = f(\phi) + g(\psi)$ , the relation of the partition function can be converted to a new one by the following form:

$$\int e^{\dots + P(\phi)} d\phi = \frac{1}{2\pi} \int d\psi d\psi_0 d\phi e^{\dots + f(\phi) + g(\psi) + i\psi_0(\phi - \psi)}, \quad (17)$$

where ... in this relation refers to other terms in the Hamiltonian. To prove the above lemma, it is enough to use the integral form of the delta function as  $\delta(\phi - \psi) = \int d\psi_0 e^{i\psi_0(\phi - \psi)}$ . In Figure (5), we also give a graphical notation for the above lemma. It shows that polynomial function  $P$  on a variable  $\phi$  can be reduced to two functions  $f$  and  $g$  on two different variables.

By lemma (17), we can show that each polynomial function can be reduced to  $\phi^4$  term. To this end, we should take three steps as follows.

**Step 1:** Consider a polynomial function as  $P(\phi) = \sum_{n=0}^N a_n \phi^n$  where if  $N$  is infinity, it can be approximated by a big number. According to the lemma (17), we can reduce this function to  $N$  functions on  $N$  variables. Therefore, we will have a new model on a different graph where on each vertex of the graph there is only a function as  $a_n \phi^n$ .

**Step 2:** Although the coefficient  $a_n$  generally can be any real number, we can reduce it to a real number which is very smaller than 1. To this end, we re-write the function  $a_n \phi^n$  in the form of summation of  $N$  terms as  $\frac{a_n}{N} \phi^n + \frac{a_n}{N} \phi^n + \dots + \frac{a_n}{N} \phi^n$ . By this fact and the lemma (17), we can reduce the function  $a_n \phi^n$  on a variable to  $N$  functions as  $\frac{a_n}{N} \phi^n$  on  $N$  different variables. It is clear that we can use a big number  $N$  so that  $\frac{a_n}{N}$  is smaller enough than 1.

**Step 3:** After the above two steps we have a new model with functions as  $t\phi^n$  where  $t$  is a very small real number. To reduce such functions to  $\phi^4$  terms, we should use a well-known relation for operators where for two operators  $A$  and  $B$  the following Identity holds:

$$e^{tA} e^{tB} e^{-tA} e^{-tB} = e^{t^2[A, B] + O(t^2)}, \quad (18)$$

where  $t$  is a real number and  $[A, B]$  is the commutation of operators  $A$  and  $B$ . If  $t$  is smaller enough than 1 we can ignore from terms with order of upper than  $t^2$  which are denoted by  $O(t^2)$ .



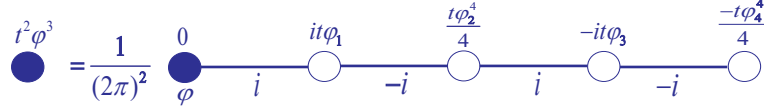


Figure 6: (Color online) The partition function of a model with the Hamiltonian  $t^2\phi^3$  on a variable in the left-hand is equal to the partition function of a new model in the right-hand where four new variables are added and the Hamiltonian of the new model has  $\phi$  and  $\phi^4$  terms.

Suppose that  $A = iP$  and  $B = \frac{Q^4}{4}$  where  $P$  and  $Q$  are momentum and position operators respectively and  $i = \sqrt{-1}$ . By the fact that  $[iP, \frac{Q^4}{4}] = Q^3$  and the Identity (18) we will have:

$$e^{itP} e^{t\frac{Q^4}{4}} e^{-itP} e^{-t\frac{Q^4}{4}} = e^{t^2 Q^3}. \quad (19)$$

Although the above identity is a relation between operators, we can convert it to an integral relation by using Eigen states of operators  $P$  and  $Q$ . To this end, we consider  $|p\rangle$  and  $|q\rangle$  as Eigen states of operators  $P$  and  $Q$  respectively where from quantum mechanics we know that  $\langle p|q\rangle = \frac{1}{\sqrt{2\pi}} e^{-ipq}$ , we suppose  $\hbar = 1$ . Then we add the Identity operator  $I = \int dp|p\rangle\langle p| = \int dq|q\rangle\langle q|$  between the operators in the relation (19). After these replacements we will have the following relation:

$$\frac{1}{(2\pi)^2} \int dq dq_1 dq_2 dp_1 dp_2 e^{itp_1} e^{t\frac{q_1^4}{4}} e^{-itp_2} e^{-t\frac{q_2^4}{4}} e^{i(qp_1 - p_1 q_1 + q_1 p_2 - p_2 q_2)} |q\rangle\langle q_2| = \int dq e^{t^2 q^3} |q\rangle\langle q|. \quad (20)$$

We can also rewrite the above relation in the following form:

$$\int dq dq_2 \{ dq_1 dp_1 dp_2 \frac{1}{(2\pi)^2} e^{itp_1} e^{t\frac{q_1^4}{4}} e^{-itp_2} e^{-t\frac{q_2^4}{4}} e^{i(qp_1 - p_1 q_1 + q_1 p_2 - p_2 q_2)} - e^{t^2 q^3} \delta(q - q_2) \} |q\rangle\langle q_2| = 0, \quad (21)$$

to hold the above identity it is necessary that the following identity holds:

$$\frac{1}{(2\pi)^2} \int dq_1 dp_1 dp_2 e^{itp_1} e^{t\frac{q_1^4}{4}} e^{-itp_2} e^{-t\frac{q_2^4}{4}} e^{i(qp_1 - p_1 q_1 + q_1 p_2 - p_2 q_2)} = e^{t^2 q^3} \delta(q - q_2). \quad (22)$$

Since variables  $p$  and  $q$  are continuous variables, we can replace them with field variables  $\phi$  and finally we have an interesting relation in the following form:

$$\frac{1}{(2\pi)^2} \int d\phi d\phi_1 d\phi_2 d\phi_3 d\phi_4 e^{it\phi_1} e^{t\frac{\phi_1^4}{4}} e^{-it\phi_3} e^{-t\frac{\phi_3^4}{4}} e^{i(\phi\phi_1 - \phi_1\phi_2 + \phi_2\phi_3 - \phi_3\phi_4)} = \int d\phi e^{t^2\phi^3}. \quad (23)$$

Suppose that in the partition function of our model there is a polynomial function as  $t^2\phi^3$ . The above relation shows that function  $t^2\phi^3$  on a variable of the model can be converted to functions  $it\phi$  and  $t\phi^4$  on four new variables. In a graphical notation it is equivalent to adding four vertices to the previous graph, see Figure (6).

So far we could convert function  $\phi^3$  to  $\phi^4$  term. By similar process we can show functions as  $t\phi^n$ , where  $t$  is very small, can also be converted to the  $\phi^4$  term. To this end, we use another commutation relation in the following form:

$$[[\frac{-P^2}{2}, \frac{Q^4}{4}], \frac{Q^n}{n}] = Q^{n+2}. \quad (24)$$

If we use this identity and apply it to relation (18) we will have:

$$e^{t^4 Q^{n+2}} = e^{t^2 [\frac{-P^2}{2}, \frac{Q^4}{4}]} e^{t^2 \frac{Q^n}{n}} e^{-t^2 [\frac{-P^2}{2}, \frac{Q^4}{4}]} e^{-t^2 \frac{Q^n}{n}}, \quad (25)$$

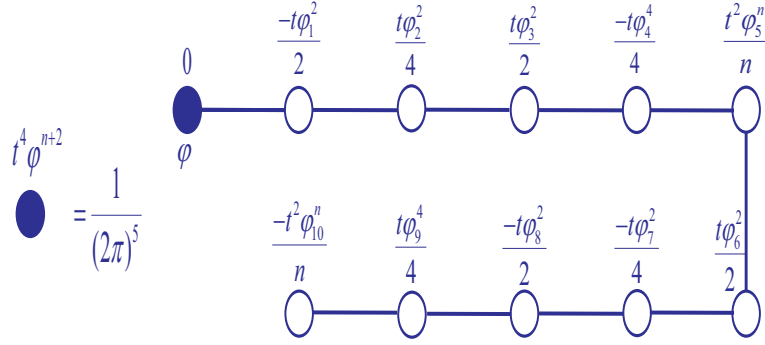


Figure 7: (Color online) The partition function of a model with the Hamiltonian  $t^4 \phi^{n+2}$  on a variable in the left-hand is equal to the partition function of a new model in the right-hand where ten new variables are added and the Hamiltonian of the new model has  $\phi^2$ ,  $\phi^4$  and  $\phi^n$  terms. By repeating this rule we can reduce polynomial functions to  $\phi^4$  term.

and by applying again to relation (18) the following identity will be derived:

$$e^{t^4 Q^{n+2}} = e^{-t \frac{P^2}{2}} e^{t \frac{Q^4}{4}} e^{t \frac{P^2}{2}} e^{-t \frac{Q^4}{4}} e^{t^2 \frac{Q^n}{n}}$$

$$e^{t \frac{P^2}{2}} e^{-t \frac{Q^4}{4}} e^{-t \frac{P^2}{2}} e^{t \frac{Q^4}{4}} e^{-t^2 \frac{Q^n}{n}}. \quad (26)$$

It is simple to find integral form of this relation by adding Eigen states of operators  $P$  and  $Q$  similar to the previous example. We show corresponding graphical notation in Figure (7). It shows that a function as  $t^4 \phi^{n+2}$  on a variable of the graph can be reduced to functions  $\phi^n$  and  $\phi^4$  and  $\phi^2$  on ten new variables. If  $n$  is a even number, it is clear that we can repeat this process to reduce power of  $\phi$  and finally we will have only terms  $\phi^2$ ,  $\phi^4$ .

Therefore if the polynomial function  $P(\phi)$  in the initial Hamiltonian (16) is an even function like  $\cos(\phi)$  for  $U(1)$  lattice gauge theory, After the above transformation we have a new model on a complex graph in the following form:

$$H = \sum_{\langle i,j \rangle} \pm i \phi_i \phi_j + \sum_i (m_i \phi_i^2 + J_i \phi_i^4), \quad (27)$$

where  $m_i$  and  $J_i$  are real numbers.

On the other hands, if  $n$  is an odd number the power of  $\phi$  is reduced to  $\phi^2$ ,  $\phi^3$  and  $\phi^4$  but we can use relation (23) to convert  $\phi^3$  to  $i\phi$  and  $\phi^4$ . Therefore, if the polynomial function  $P(\phi)$  in relation (16) is an odd function, after transformation we have a new model on a complex graph in the following form:

$$H = \sum_{\langle i,j \rangle} \pm i \phi_i \phi_j + \sum_i (i h_i \phi_i + m_i \phi_i^2 + J_i \phi_i^4), \quad (28)$$

where  $h_i$ ,  $m_i$  and  $J_i$  are real numbers.

## 5 Transformation for decreasing dimension

In the previous sections we showed that one can map the discrete version of classical field theories and  $U(1)$  lattice gauge theories on arbitrary graphs to a  $\phi^4$  field model with the same partition function by

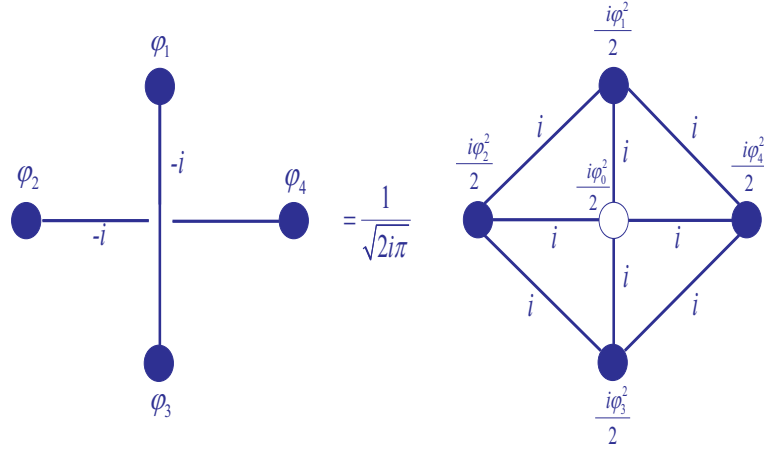


Figure 8: (Color online) The partition function of a model in presence of crossing between two edges in the left-hand is equal to partition function of a new model in the right-hand where a new variable is added to annihilate crossing. In this way the initial graph is converted to a planar graph.

adding many new vertices on the initial lattice. It is clear that the final graph after adding new vertices is a complex graph and we have not a specific form for it. Specially such a graph is not a planar graph because the initial model may be in any dimension.

In this section we give another transformation to decrease dimension of the model to two. To this end, we consider a complex graph which is derived after transformations in the previous sections. If we flatten this graph on a two-dimensional page, we will see many crossings between edges of the graph which shows that graph is not planar. A crossing of the graph has been shown in Figure (8) where there are four variables  $\phi_1, \phi_2, \phi_3$  and  $\phi_4$  which interact with each other as  $-i\phi_1\phi_3 - i\phi_2\phi_4$ . Therefore, in the partition function we have an integral in the following form:

$$I = \int d\phi_1 d\phi_2 d\phi_3 d\phi_4 e^{-i\phi_1\phi_3 - i\phi_2\phi_4}. \quad (29)$$

We define a new variable  $\phi_0$  and insert it into the place of crossing of the graph. Then we use a simple identity in the following form:

$$\int d\phi_0 e^{\frac{i}{2}(\phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4)^2} = \sqrt{2i\pi}. \quad (30)$$

We multiply this factor to the phrase (29) and we will have:

$$I = \frac{1}{\sqrt{2i\pi}} \int d\phi_0 d\phi_1 d\phi_2 d\phi_3 d\phi_4 e^{\frac{i}{2}(\phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4)^2 - i\phi_1\phi_3 - i\phi_2\phi_4}. \quad (31)$$

After simplification we will reach to a new form of the phrase (29) in the following form:

$$I = \frac{1}{\sqrt{2i\pi}} \int d\phi_0 d\phi_1 d\phi_2 d\phi_3 d\phi_4 e^{i(\phi_1\phi_2 + \phi_2\phi_3 + \phi_3\phi_4 + \phi_4\phi_1) + i\phi_0 \sum_{i=1}^4 \phi_i + \frac{i}{2} \sum_{i=0}^4 \phi_i^2}, \quad (32)$$

transformation from phrase (29) to phrase (32) has been shown graphically in Figure (8). It shows that crossing in the initial graph can be annihilated by adding a new variable and some changes in the connection pattern of the graph.

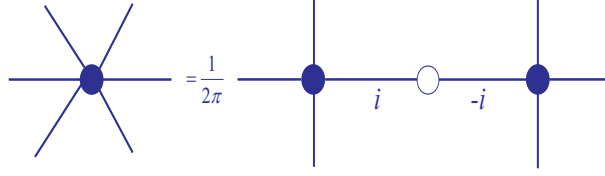


Figure 9: (Color online) The partition function of a model in the left-hand is equal to the partition function of a new model in the right-hand where the initial variable is splitted to two new variables. In this way six connections of the initial variable on the left side are also splitted to three links of new variables in the right-hand. If the degree of the original variable is upper than six, it is again possible to reduce the degree of all vertices to four by repeating this rule.

Finally, after annihilating all crossings in the initial graph we have a new model on a planar graph in the following form:

$$H = \sum_{\langle i,j \rangle} \pm i \phi_i \phi_j + \sum_i (i h_i \phi_i + m_i \phi_i^2 + J_i \phi_i^4), \quad (33)$$

where  $h_i$  is a real number which was necessary for odd functions. Furthermore, Since complex function  $\frac{i}{2} \phi_i^2$  is necessary for reduction of dimension,  $m_i$  can be a complex number and  $J_i$  is still a real number.

## 6 Matching to the 2D square lattice

After transformation of the initial model to a  $\phi^4$  theory on a planar graph in the previous section, we are ready to complete result by matching the model on a 2D square lattice. The Hamiltonian (33) is defined on a complex planar graph. In order to match such a graph to a square lattice, there are three points which should be considered. We explain these points in three steps:

### Step 1: Reducing the degree of vertices

The most important problem for matching the graph on a square lattice is to reduce the degree of vertices. In a square lattice, the degree of each vertex is four while in the our model (33) the degree of each vertex may be any arbitrary numbers. We use the lemma (17) to reduce the degree of a vertex. For example, according to lemma (17) as it is shown in Figure (9), we can split a vertex which has six links with other vertices of the lattice to two new vertices where there are only three connections for each one of them. For the vertices of the graph with higher degrees it is enough to repeat this rule for more times to finally reduce the degree to four.

### Step 2: Link insertion

Second problem is that we should match all edges of the final graph to edges of the square lattice. To this end, it is necessary to insert many vertices on the edges of the graph to match it to the vertices of the square lattice. We use a simple rule for adding vertices on any links of the graph in the following form:

$$\int d\phi_1 d\phi_2 e^{i\phi_1 \phi_2 - \frac{i}{2} \phi_1^2 - \frac{i}{2} \phi_2^2} = \frac{1}{\sqrt{2\pi i}} \int d\phi_0 d\phi_1 d\phi_2 e^{i\phi_0 \phi_1 - i\phi_0 \phi_2 + \frac{i}{2} \phi_0^2}. \quad (34)$$

This relation has been shown in Figure (10) in a graphical notation. To prove this relation it is enough to complete the square as  $(\phi_0 - (\phi_2 - \phi_1))^2$  in the right-hand of the relation and perform integration on  $\phi_0$ . In this way, we can add enough number of vertices to completely match the graph to the square lattice.

### step 3: Face insertion

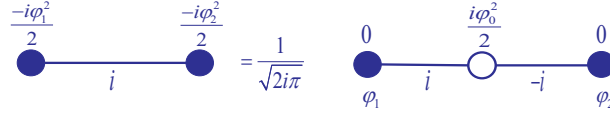


Figure 10: (Color online) Adding a new variable to a link in the left-hand changes the partition function only by a complex factor  $\frac{1}{\sqrt{2i\pi}}$  and also changes polynomial functions on variables in the right-hand.

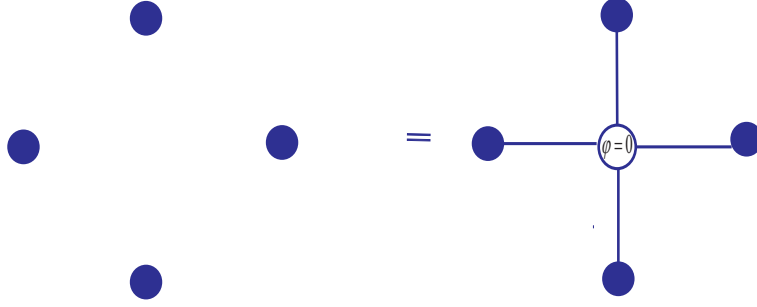


Figure 11: (Color online) Adding a new variable  $\phi$  which is fixed to zero to each empty point of the lattice does not change the partition function.

Last step for complete matching to a square lattice is that there is also some empty points on square lattice which should be filled by new variables. To this end, we can add a new variable  $\phi$  in each point of the square lattice and fix its value to zero. As it is shown in Figure(11), such a transformation does not lead to any extra terms in the partition function.

## 7 completeness of 2D classical $\phi^4$ theory

In the previous sections we gave few steps to convert a general field theory to a new model on a square lattice in the following form:

$$H = \sum_{\langle i,j \rangle} \pm i \phi_i \phi_j + \sum_i (i h_i \phi_i + m_i \phi_i^2 + J_i \phi_i^4), \quad (35)$$

by a simple transformation it is possible to change this model to standard  $\phi^4$  field theory. To this end, we use this fact that  $i \phi_i \phi_j = -\frac{i}{2}(\phi_i - \phi_j)^2 + \frac{i}{2}\phi_i^2 + \frac{i}{2}\phi_j^2$  and after replacement of this relation to the above Hamiltonian we will have:

$$H = \sum_{\langle i,j \rangle} \pm \frac{1}{2}(\phi_i - \phi_j)^2 + \sum_i (i h_i \phi_i + m_i \phi_i^2 + J_i \phi_i^4), \quad (36)$$

where we absorb  $\pm \frac{i}{2}\phi_i^2$  in the term  $m_i \phi_i^2$ . In this way, our complete model is a classical  $\phi^4$  field model on a square lattice where coupling constants of the model can be complex numbers but  $J_i$  is a real number.

There is also another important point that we should consider about our results. In fact for transformation of the initial model to a complete model as (36) we added many new vertices to the initial lattice. such a work would not be efficient if the number of added vertices were as an exponential function of initial number of vertices. But it is simple to show that in all transformations that we gave,

the number of added vertices were limited so that total number of added vertices certainly will be a polynomial function of initial number of vertices. Furthermore, we emphasize that although some of our steps in unification of various models in a  $\phi^4$  field theory include approximations, we could perform the unification with arbitrary precision.

## 8 Discussion

This problem that a single theory of physics is enough for explaining other physical theories is a big challenge that is being followed by scientists in various fields of physics. In this paper we studied the completeness of 2D classical  $\phi^4$  field theory which is interested in statistical physics as well as field theory. We gave a step by step proof which shows how a discrete version of classical field theories and also  $U(1)$  lattice gauge theories on arbitrary graphs with arbitrary dimensions can be converted to a 2D classical  $\phi^4$  field theory with the same partition function. Since our result was derived by a purely mathematical method, quantum information theory which already played a key role was absent in our proof. However, we believe that our approach has this advantage that due to simplicity and generality, it leads to a general approach which can be used by various communities of physicists for future researches on similar problems.

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